11

Boundaries in Visualizing Mathematical Behaviour

Andrew Francis Hare
Saint Mary's University

It is surprising to students to learn that a natural combination of simple functions, the function sin(1/x), exhibits behaviour that is a great challenge to visualize. When x is large the function is relatively easy to draw; as x gets smaller the function begins to behave in an increasingly wild manner. The sin(1/x) function can serve as one of their first counterexamples, helping them to better appreciate the tamer functions that they normally encounter. I see three boundaries here. First, a boundary erected by mathematicians between 'nice' versus 'wild' functions - captured for example by the concept of continuity. Second, a boundary between those functions that are most often studied in calculus and pre-calculus classrooms, and those that are more rarely looked at. Third, the boundary between the drawable and the undrawable. In this example, we can witness this last boundary first-hand even as we attempt to sketch the curve. Yet, we can also continue the visualization in our mind's eye beyond what we can represent on paper.

Introduction

It is possible to advance quite far in one's mathematics education without encountering the function sin(1/x). I describe it as an "encounter" because it felt that way to me when I read Hardy's *A Course of Pure Mathematics* (1908) and reached Exercise 6 of Examples XV in Section 28: "Draw the graph of sin(1/x)." I remember distinctly how strange it was at first to realize just how wild this function's behaviour was near the origin. Unlike so many other graphs of functions of the sort that are treated either in upper-secondary school courses or beginning-undergraduate courses, this function seemed to display an ever-concentrating vigour and vitality in an ever-narrowing neighbourhood. By stark contrast, the 'nice' functions I had spent most of my time learning about looked simpler and simpler the more I zoomed in my visual field on any particular point on the graph. I thought I was familiar with sin(x) and with 1/x, but a simple combination of them held surprises for me, and I felt I was meeting with a stranger.

The drawing I created on my page and the drawing on the following page in Hardy (1908) interested me greatly. These drawings captured some of the behaviour of the function but left out a region near x=0 that was impossible to render accurately. Hardy made the choice to leave the region blank; I continued drawing oscillations until I had nearly filled in with ink the region near x=0 and between -1 and 1. The latter choice is the one that Google makes, for what it is worth, when you search for sin(1/x). The former choice is more honest, in the sense that it does not draw anything that is not accurate. In both cases, it seemed to me, the mind

needed to continue an internal visual representation of an external visual representation that could not be continued.[1]

I have taught introductions to calculus to various groups of students: Grade 12 students, first-year undergraduate students in the life sciences, first-year undergraduate students in the physical sciences, in-service teachers who are taking a certificate program in mathematics. To all of these students I have introduced the graph of sin(1/x). I believe that considering this curve and contemplating its properties can provide valuable insights to any student of mathematics. I explore some of these insights below; images of sin(1/x) appear just before the end.

I emphasize to students the following important feature of our task: we are graphing the composition of two functions that we are already well acquainted with. The function 1/x exhibits behaviour familiar from many situations, including, for example, pressure-volume dependence in ideal gases or price-quantity demanded in classical economics. I want students to realize that out of well-known, relatively familiar ingredients we can very quickly encounter an object that is unfamiliar and strange. We consider sin(x) and we consider 1/x: Are we to pass over in silence the natural question of sin(1/x)? I believe that some students consciously notice these silences in the curriculum and that more students can sense but can't articulate that they are being shepherded through a field of mathematical objects and steered clear of any dangers by the helpful teacher. This perhaps leaves them with little confidence that they could steer themselves.

There might be the rejoinder, "Natural question? Natural to whom?" But if this is a suggestion that to many students graphing sin(1/x) is not a natural question, then this is even more of a reason to treat this function and to suggest to students that it is a natural question to ask. As people exploring a mathematical terrain, they have the right to look into every nook and cranny, and to try every combination of operations that they know about in order to seek something new, or in order to completely understand inside and out the tools they have at hand.

It becomes clear to the class as they discuss sin(1/x), after having drawn sin(x) carefully, that they need to appreciate the behaviour of 1/x a little more clearly. This is an opportunity to impress upon students how a simple idea, one that they have been familiar with for so long, still contains surprises. For many students, it is important to see what 1/x really does, as if for the first time, to make it new. They have learned a verbal formula ('all you do is take the reciprocal' or 'just flip it'). I encourage them to draw a horizontal line representing the real numbers. The activity now is to take a generic sample of points, to plot them on the line, to plot where 1/x takes those points, and, finally, to connect these two points – the input and output – with an arrow. It becomes clear that the points x=1 and x=-1 are special values. Points to the right of x=1 get sent to points in the interval (0, 1), but, more importantly, the students' resulting picture will help convince them that x=10 and x=50, for example, though separated by a gap of 40, get sent to points that are only .08 apart, and I ask them to calculate such an example together as a class.

I am not trying to describe a method whereby a difficult concept is rendered easy to attain at first attempt. Graphing sin(1/x) simply is challenging, and yet there are clear rewards for struggling through understanding its behaviour. I will list a few of these rewards later, but for now I want to name one: many other problems will seem much easier, not only by comparison, but because the effort involved in coordinating together the various ideas here will make all of the individual ideas much more clearly understood. As Thurston (1990) notes:

> Mathematics is amazingly compressible: you may struggle a long time, step by step, to work through some process or idea from several approaches. But once you really understand it and have the mental perspective to see it as a whole, there is often a tremendous mental compression. You can file it away, recall it quickly and completely when you need it, and use it as just one step in some mental process. The insight that goes with this compression is one of the real joys of mathematics. (p. 846)

It is important for students to have the opportunity in their mathematical education to have experiences like the one Thurston describes. It is, of course, tempting to skip the sin(1/x) struggle entirely, but perhaps a great deal would be lost by sheltering students from this struggle, as if one did not quite believe in them.

Although some students are able to proceed at

[1] For discussions of visual representations in mathematics see Arcavi (2003) and Dreyfus (1991).

this point on their own, other students will need some supportive hints, starting with the briefest of suggestions. I usually suggest starting the graph of sin(1/x) when x is large. Already this is an unusual aspect of graphing for many students. Getting over this hurdle (that it is not necessary to graph a function 'from left to right'), is in itself conceptual reward enough for tackling this problem. We can begin a graph anywhere we like, move in any direction we like, if we have knowledge that takes us there.

At some point a student will mention that, because sin(x) oscillates up and down an infinite number of times as x travels from 2π to infinity, sin(1/x) will oscillate up and down an infinite number of times as x travels from $1/(2\pi)$ to zero. Here is an amazing compression indeed!

I have mentioned the challenging aspect of graphing this function. It is time now to address the fact that this function is, at some level, a disturbing, unsettling, frightening function. I have yet to have a student run screaming from the room, but there is no question that the consideration of this function does provoke emotional 'turning away' reactions at the same time that it fascinates. Opportunities for strong aesthetic responses in mathematics education ought to be cultivated. If some mathematicians had negative reactions to encountering such functions it is understandable that students will. It is useful to sketch a little bit of this history at the anecdotal level so that students appreciate that their reactions are natural. For many students, like mathematicians before them, their initial aesthetic assessment of distaste is replaced by enjoyment and appreciation (Sinclair, 2004). Students can feel a kinship with their predecessors.

I emphasize taking it slow so that the wild oscillations in a narrow region do not overwhelm the students with their complexity and strangeness. This function is one of the first in a series of so-called pathological functions that were considered first by mathematicians in the 19th century and early 20th century (Tall, 1982; Tall, 1991). One of the results of these encounters is that mathematicians defined the notion of continuity. Continuous functions do not display the bizarre behaviour seen above. The place in the curriculum where continuity arguably makes its most important appearance is in the hypothesis of the Fundamental Theorem of Calculus. If we are to be honest with students, we need to show them examples of functions that are not continuous so that their intuition about different sorts of functions can be appropriately generalized.

When it comes time to draw the picture that helps motivate the justification of the Fundamental Theorem of Calculus, one inevitably draws a 'general' function, labels two points on the x-axis nearly side by side, and points to the picture in order to show that the shape of the region above this narrow interval is very nearly a rectangle with a width and height that are easy to determine. With no counterexample in the offing, many students might nod along as if all of this were obvious. With sin(1/x) in the background, students can realize that it is not true that we can expect such nice behaviour from all functions. Their experience with the function discussed in this paper has changed their very notion of what a 'generic' function can look like. They therefore realize there is something special about functions that do enjoy this behaviour. Continuity becomes a prize that is appreciated for what it is, rather than an empty definition that 'every' function satisfies anyway. It is important for students to have at the ready a selection of examples to test new theoretical results on (Goldenberg & Mason, 2008; Sinclair, Watson, Zazkis, & Mason, 2011).

The following images are examples of sin(1/x) (with $x > 0$). The first image is very similar to what appeared in Hardy (1908):

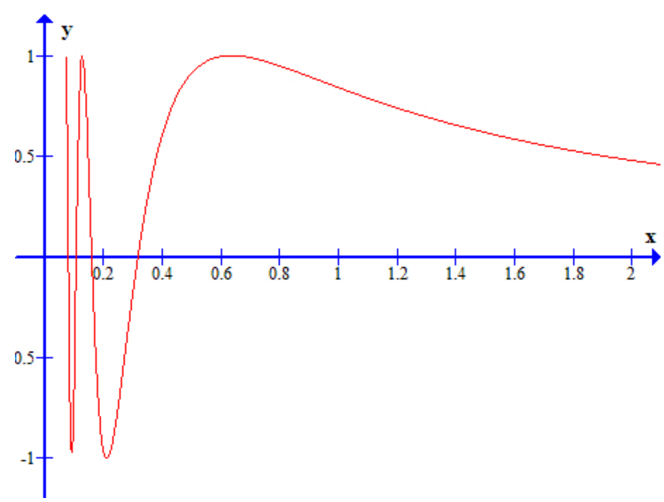

Figure 1

sin(1/x)

The curve continues to oscillate as it approaches the y-axis; in these images we cut off these (infinitely many) oscillations. To see a few more of them, we move closer:

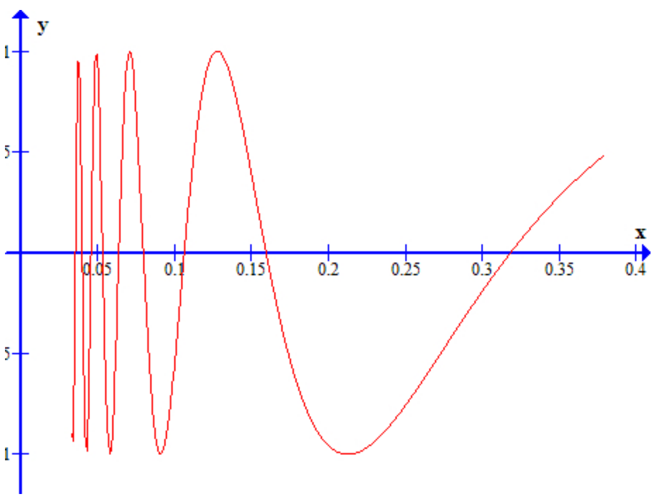

Figure 2

sin(1/x)

And still closer:

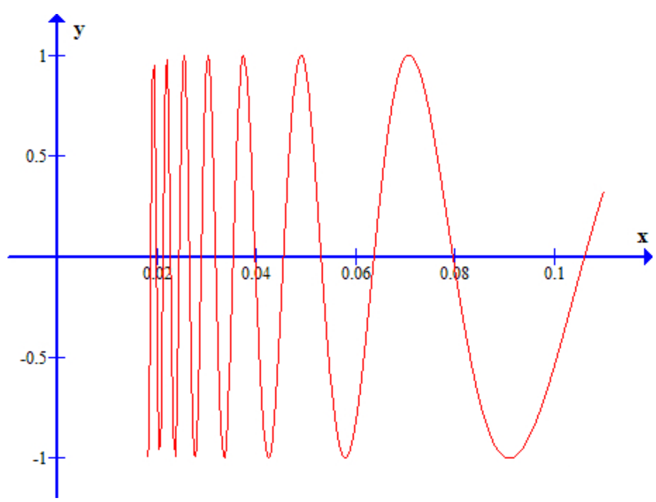

Figure 3

sin(1/x)

Note that all the peaks of these oscillations should equal 1, and all of the minimums of the valleys should equal -1. The excellent open-source graphing program Graph (created by Ivan Johansen) stumbles a little here (some peaks seem a little short, for example). This is no criticism of the program: **all** numerical attempts to graph sin(1/x) must inevitably stumble. Also, the curve should be smooth, and not 'pixelated'. For some students, witnessing these flaws and comparing them to the perfect image in their mind is a key moment in their mathematics education.

Understanding the graph of sin(1/x) helps students understand the power of the visual representation of a function as well as appreciate its limitations at exactly the same time. They are offered a concrete picture of the boundary between the drawable and undrawable. It helps them appreciate the lasting power and interest of the single concrete example. The process of understanding the graph allows them to confront and overcome negative reactions to strange and unfamiliar behaviour by reinterpreting them in positive ways. The students take part in a process that is a miniature encapsulation of the process by which new mathematical understandings are born and communicated.

References

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics, 52*, 215-241.

Dreyfus, T. (1991). On the status of visual reasoning in mathematics and mathematics education. In F. Furinghetti (Ed.), *Proceedings of the 15th PME International Conference*, *1*, 33-48.

Goldenberg, P. & Mason, J. (2008). Shedding light on and with example spaces. *Educational Studies in Mathematics*, *69*, 183-194.

Hardy, G.H. (1908). *A course of pure mathematics.* (10th ed., 1960). Cambridge: CUP.

Sinclair, N. (2004). The roles of the aesthetic in mathematical inquiry. *Mathematical Thinking and Learning*, *6*, 261-284.

Sinclair, N., Watson, A., Zazkis, R., & Mason, J. (2011). The structuring of personal example spaces. *The Journal of Mathematical Behavior, 30*, 291-303.

Tall, D. (1982). The blancmange function, continuous everywhere but differentiable nowhere. *Mathematical Gazette*, *66*, 11-22.

Tall, D. (1991). Intuition and rigour: The role of visualization in the calculus. In W. Zimmerman & S. Cunningham (Eds.), *Visualization in teaching and learning mathematics* (pp. 105-119). MAA Notes No. 19.

Thurston, W.P. (1990). Mathematical education. *Notices of the American Mathematical Society, 37*, 844-850.

Biography

Andrew Hare is a Lecturer in the Mathematics & Computing Science Department at Saint Mary's University. He is interested in mathematics education, mathematics and language, the popularization of mathematics, and the philosophy of mathematical practice.